Quantum Systems from Ground up

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What sets QM apart

- Classical physics aims at an objective description description of the real world
- Quantum mechanics quantifies our *knowledge*you want of the world
  - It recognises that our knowledge derives from measurements
  - And it acknowledges that to measure is inevitably to disturb

To measure is to disturb

- You want to image a virus?
- Head to SLAC for large flux of coherent short λ radiation

- Virus scatters X-rays, which form diffraction pattern
- From pattern the image is computed

To measure is to disturb

- You want to image a virus?
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- Virus scatters X-rays, which form diffraction pattern
- From pattern the image is computed
- The sample is totally destroyed by the pulse:
  - to image can be to destroy

Probabilities & amplitudes

- QM $\Rightarrow$ the statistics of repeated measurement via $P(q)$
  - $P(q)$ used widely in the physical and social sciences
  - But QM is unique in how it computes $P(q)$
- QM first computes complex $A(q)$ and uses $P(q) = |A(q)|^2$
- The quantum amplitude contains more information than $P(q)$: $A(q) = \sqrt{P(q)}\,\mathrm{e}^{\mathrm{i}\phi}$

2-state systems

- Simplest system is a '2-state system' a.k.a. 'qubit'
- '2-state' because ideal measurements have 2 outcomes:
  - call them up/down, or in/out, or $\uparrow\downarrow$
- We focus on qubits but results generalise

States & state vectors

- In state $|\psi\rangle$ the qubit has an amplitude a to be measured up, and amplitude b to be measured down
- So $|\psi\rangle \leftrightarrow (a, b)$ a pair of complex numbers
- So qubit states are complex 2-vectors
  - points $\mathbf{r}$ in real space are real 3-vectors (x, y, z)

Measurement & wavefunction collapse

- QM $\Rightarrow$ statistics of measurements
  - But it doesnt want to engage with the defects of our equipment, the cack-handedness of my practical partner!
- So it deals with ideal (reproducible) measurements
  - After measuring Q with result q, a 2nd measurement is certain to yield q
- The state in which q is certain is called $|q\rangle$
  - An ideal measurement of Q jogs the system into $|q\rangle$
- For qubits we write

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \xrightarrow{\text{measure}} \begin{cases} |\uparrow\rangle & \text{with prob } |a|^2 \\ |\downarrow\rangle & \text{with prob } |b|^2 \end{cases}$$

$$(a,b) = a(1,0) + b(0,1) \xrightarrow{\text{measure}} (1,0) \text{ or } (0,1)$$

- 'we call this collapse of the wavefunction'

Spin-half

- Electrons, protons, neutrons, etc are tiny gyros
  - They spin at a constant rate, but you can change the direction of spin
- If you measure any component of angular momentum S_i, the only answers are $\pm\frac{1}{2}\hbar$
- So these spins are perfect qubits

spin-half

- The states $|\uparrow z\rangle$ & $|\downarrow z\rangle$ that the spin is jogged into on measuring S_z are naturally different from the states $|\uparrow x\rangle$ & $|\downarrow x\rangle$ it's jogged into when you measure S_x
- Measurements are often incompatible because the set $\{|q\rangle\}$ is particular to Q

$$|\psi\rangle \xrightarrow{S_z;|a|^2} |\uparrow z\rangle \xrightarrow{S_x;\frac{1}{2}} |\uparrow x\rangle \xrightarrow{S_z;\frac{1}{2}} |\downarrow z\rangle$$

  - 'uncertainty principle'

Bras

- It's important to have a tool to extract from $|\psi\rangle$ the amplitude a_n to measure, say, energy E_n
- Let the states with definite energy be denoted $|E_i\rangle$
- Then we define a matching set of functions $\langle E_i|$ by the equation $\langle E_i|E_j\rangle = \delta_{ij}$
- This rule & the linearity of the functions enables us to evaluate the functions on any state:

$$\langle E_i|\psi\rangle = \langle E_i|\Big(\sum_j a_j|E_j\rangle\Big) = \sum_j a_j\delta_{ij} = a_i$$

- $\langle\phi|\psi\rangle$ is a bra-ket (PAM Dirac)

Observables & operators

- With bras we can define an operator $H = \sum_n |E_n\rangle E_n \langle E_n|$
- H turns kets into kets: $H|\psi\rangle = \sum_n |E_n\rangle E_n \langle E_n|\psi\rangle$
- We have $H|E_m\rangle = E_m|E_m\rangle$, an eigenvalue equation
  - so H *is* the operator with eigenvalues E_n & eigenkets $|E_n\rangle$
- By this procedure every observable yields an operator
- None is as important as the Hamiltonian H on account of the Time Dependent Schrödinger Equation (TDSE)

$$\mathrm{i}\hbar \frac{\mathrm{d}|\psi\rangle}{\mathrm{d}t} = H|\psi\rangle$$

- Now consider a system with 2 qubits
- Homework: show that any 2-qubit state can be expressed as a linear combination of

$$|\uparrow\rangle|\uparrow\rangle,\ |\uparrow\rangle|\downarrow\rangle,\ |\downarrow\rangle|\uparrow\rangle,\ |\downarrow\rangle|\downarrow\rangle$$

  or by shorthand $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$
  - In $|\uparrow\uparrow\rangle$ both qubits are certain to return up, etc
- That is, always $|\psi\rangle = c_{\uparrow\uparrow}|\uparrow\uparrow\rangle + c_{\uparrow\downarrow}|\uparrow\downarrow\rangle + c_{\downarrow\uparrow}|\downarrow\uparrow\rangle + c_{\downarrow\downarrow}|\downarrow\downarrow\rangle$

Entanglement

- If $|\psi\rangle = |\uparrow\rangle\left(|\uparrow\rangle + |\downarrow\rangle\right)/\sqrt{2}$
  - $S_z^{(1)}$ is certain to return $\uparrow$
  - $S_z^{(2)}$ has $p = \frac{1}{2}$ to return either $\uparrow$ or $\downarrow$
  - Measuring $S_z^{(1)}$ doesn't change $|\psi\rangle$
  - Measuring $S_z^{(2)}$ collapses

$$|\psi\rangle \xrightarrow{S_z^{(2)}} \begin{cases} |\uparrow\rangle|\uparrow\rangle \text{ with } p = \frac{1}{2} \\ |\uparrow\rangle|\downarrow\rangle \text{ with } p = \frac{1}{2} \end{cases}$$

    so whatever the outcome, $S_z^{(1)}$ still certain ton yield $\uparrow$correlated/
  - The odds on the remaining measurement are unchanged by measuring one spin:
  - the spins aren't *correlated*/entangled

Entanglement

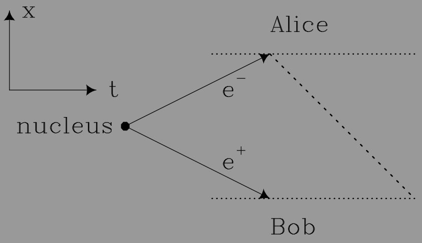
- With $\psi = (|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)/\sqrt{2}$ it's different:
  - $$|\psi\rangle \xrightarrow{S_z^{(1)}} \begin{cases} |\uparrow\rangle|\downarrow\rangle \text{ with } p = \frac{1}{2} \\ |\downarrow\rangle|\uparrow\rangle \text{ with } p = \frac{1}{2} \end{cases}$$
  - After the first measurement the result of the second becomes certain
  - The spins are entangled
- These are examples of a general result
  - Systems are correlated/entangled if and only if their state in not a product state $|\phi\rangle|\phi'\rangle$

EPR experiment

- The entangled 2-spin state

$$|0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

has zero spin because the two gyros are anti-aligned
- $|0\rangle$ provides the cleanest version of an scenario first considered by Einstein, Podolsky & Rosen (1935)



- Alice: $S_z \longrightarrow |\uparrow\rangle$: $|\psi\rangle \longrightarrow |\uparrow\rangle|\downarrow\rangle$
- Alice: $S_x \longrightarrow$ either $|\uparrow x\rangle|\downarrow x\rangle$ or $|\downarrow x\rangle|\uparrow x\rangle$

EPR experiment

- EPR wrote
  - 'as a consequence of two different measurements performed upon the first system, the second system may be left in states with two different wavefunctions. On the other hand, since at the time of measurement the two systems no longer interact, no real change can take place in the second system...Hence two different wavefunctions can be assigned to the same reality'

Resolution of EPR experiment

- Yes two states one reality because state reflects knowledge
- Until Alice acts, the positron doesn't have a state
- By measuring her electron Alice gains information about the positron needed to assign it a state
- Alice changes the *pair's* state by jogging it
- She doesn't discover the orientation of the electron before she jogged it, so she doesn't discover the positron's orientation
- Bob's measurement is not affected by Alice's though it is *correlated* with hers

Summary

- QM makes statistical predictions from complex amplitudes
- States are defined by sets of complex numbers
- Measurements jog the system into specific states
- Observables are associated with operators
- When systems are combined, correlations are likely
- Subsystems have wavefunctions only when uncorrelated